Tutorial 12

April 21, 2016

1. **Derive the representation formula for harmonic functions in two dimensions**

   Let $u \in C^1(\bar{D}) \cap C^2(D)$ be a harmonic function on D. Let $\mathbf{x}_0 \in D$, then

$$u(\mathbf{x}_0) = \frac{1}{2\pi} \int_{\partial D} \left[u(\mathbf{x}) \frac{\partial}{\partial n} \log |\mathbf{x} - \mathbf{x}_0| - \frac{\partial u}{\partial n} \log |\mathbf{x} - \mathbf{x}_0| \right] ds$$

   Solution: Let D_ϵ be the region D with a ball (of radius ϵ and the center $\mathbf{x}_0$) excised. For simplicity let $\mathbf{x}_0$ be the origin and set $r = \sqrt{x^2 + y^2}$. By the Green's second identity,

$$\int_{\partial D_\epsilon} \left[u \cdot \frac{\partial \log r}{\partial n} - \frac{\partial u}{\partial n} \cdot \log r \right] dS = 0.$$

   Noting that ∂D_ϵ consists of two parts and on $\{|\mathbf{x}| = r = \epsilon\}$, $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$, we have

$$\int_{\partial D} \left[u \cdot \frac{\partial \log r}{\partial n} - \frac{\partial u}{\partial n} \cdot \log r \right] dS = \int_{r=\epsilon} \left[u \cdot \frac{\partial \log r}{\partial r} - \frac{\partial u}{\partial r} \cdot \log r \right] dS, \quad \text{for } \forall \epsilon > 0. \tag{1}$$

   Now the right side of the identity equals

$$\frac{1}{\epsilon} \int_{r=\epsilon} u dS - \log \epsilon \int_{r=\epsilon} \frac{\partial u}{\partial r} dS = 2\pi \bar{u} - 2\pi\epsilon \log \epsilon \overline{\frac{\partial u}{\partial r}},$$

   where $\overline{u}$ denotes the average value of u on the circle $\{r = c\}$, and $\overline{\frac{\partial u}{\partial r}}$ denotes the average value of $\frac{\partial u}{\partial r}$ on this circle. Since u is continuous and $\frac{\partial u}{\partial r}$ is bounded, we have

$$2\pi\overline{u} - 2\pi\epsilon \log \epsilon \overline{\frac{\partial u}{\partial r}} \to 2\pi u(0) \quad \text{as } \epsilon \to 0.$$

   So let ϵ tend to 0 in identity (1) and then we can obtain the representation formula.

2. **Theorem 2 on P181**

   The solution of the problem

$$\Delta u = f \quad \text{in } D \quad u = h \ \text{ on } \partial D$$

   is given by

$$u(\mathbf{x}_0) = \iint_{\partial D} h(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}_0)}{\partial n} dS + \iiint_D f(\mathbf{x}) G(\mathbf{x}, \mathbf{x}_0) d\mathbf{x}$$

   Solution: Let $v(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}, \mathbf{x} \neq \mathbf{x}_0$, then $\Delta v(\mathbf{x}) = 0, \mathbf{x} \neq \mathbf{x}_0$.

   Let D_ϵ be the region D with a ball (of radius ϵ and the center $\mathbf{x}_0$) excised.

   Applying Green's Second Identity to v and u on D_ϵ, we have

$$\iiint_{D_\epsilon} -vf d\mathbf{x} = \iiint_{D_\epsilon} u\Delta v - v\Delta u d\mathbf{x} = \iint_{\partial D_\epsilon} \left[u \cdot \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} \cdot v \right] dS$$

Noting that ∂D_ϵ consists of two parts and on $\{|\mathbf{x}-\mathbf{x}_0| = r = \epsilon\}$, $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$, we have

$$\iint_{r=\epsilon} u\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n}vdS = -\iint_{r=\epsilon} u\frac{\partial v}{\partial r} - \frac{\partial u}{\partial r}vdS = -\frac{1}{4\pi\epsilon^2}\iint_{r=\epsilon} udS - \frac{1}{4\pi\epsilon}\iint_{r=\epsilon}\frac{\partial u}{\partial r}dS = -\bar{u} - \epsilon\overline{\frac{\partial u}{\partial r}}$$

where $\overline{u}$ denotes the average value of u on the sphere $\{r = c\}$, and $\overline{\frac{\partial u}{\partial r}}$ denotes the average value of $\frac{\partial u}{\partial r}$ on this sphere. Since u is continuous and $\frac{\partial u}{\partial r}$ is bounded, we have

$$-\overline{u} - \epsilon\overline{\frac{\partial u}{\partial r}} \to -u(\mathbf{x}_0) \quad \text{as } \epsilon \to 0.$$

So let ϵ tend to 0 and then we have

$$\iiint_D -vfd\mathbf{x} = \iint_{\partial D}\left[u\cdot\frac{\partial v}{\partial n} - \frac{\partial u}{\partial n}\cdot v\right]dS - u(\mathbf{x}_0) \tag{2}$$

Suppose $G(\mathbf{x},\mathbf{x}_0)$ is the Green's function for $-\Delta$, then $H = G - v$ is a harmonic function on D, and $G = 0$ on ∂D. Applying the second Green's Identity to u and H on D, we have

$$\iiint_D -Hfd\mathbf{x} = \iiint_D u\Delta H - H\Delta ud\mathbf{x} = \iint_{\partial D}\left[u\cdot\frac{\partial H}{\partial n} - \frac{\partial u}{\partial n}\cdot H\right]dS \tag{3}$$

Adding (2) and (3) and using $G = H + v$ in D,$G = 0$ on ∂D , we get

$$\iiint_D -Gfd\mathbf{x} = \iint_{\partial D}\left[u\cdot\frac{\partial G}{\partial n} - \frac{\partial u}{\partial n}\cdot G\right]dS - u(\mathbf{x}_0) = \iint_{\partial D} h\frac{\partial G}{\partial n}dS - u(\mathbf{x}_0)$$

That is,

$$u(\mathbf{x}_0) = \iint_{\partial D} h\frac{\partial G}{\partial n}dS + \iiint_D Gfd\mathbf{x}$$